The fluctuation-dissipation theorem in the framework of the Tsallis statistics

1994 J. Phys. A: Math. Gen. 27 3663

(http://iopscience.iop.org/0305-4470/27/11/016)

J. Phys. A: Math. Gen. 27 (1994) 3663–3670. Printed in the UK

The fluctuation–dissipation theorem in the framework of the Tsallis statistics

A Chame and E V L de Mello

Instituto de Fisica, Universidade Federal Fluminense, Outeiro São João Batista s/nº—Centro 24.030.130—Niterói, RJ, Brazil

Received 28 September 1993

Abstract. In the framework of a generalized statistical mechanics introduced recently by Tsallis, we derive a generalized form of the fluctuation–dissipation theorem, which expresses a relation between extended susceptibilities and equilibrium fluctuations. To achieve this, we consistently propose a generalized functional form for the instantaneous distribution function. The present theorem recovers as particular cases the corresponding generalized relations already obtained for the specific heat in terms of the generalized energy fluctuations and for the susceptibility of a magnetic system under the action of a uniform magnetic field.

1. Introduction and general considerations

A generalized entropy has been recently introduced by Tsallis [1]. This entropy is given by the expression

$$S_q \equiv k \frac{1-\Sigma_{i=1}^{N} p_i^q}{q-1} \tag{1}$$

where k is a conventional positive constant, $q \in \mathbb{R}$ characterizes the generalization, p_i is the probability of occurrence of the ith microstate of the system and N is the total number of such microstates. The $q \to 1$ limit yields the standard Shannon entropy

$$S = -k_B \sum_{i=1}^{N} p_i \ln(p_i). \tag{2}$$

Preserving the standard variational principle, Tsallis [1] established the microcanonical and canonical generalized distributions. For the microcanonical ensemble, i.e. in the case of equiprobability ($p_i = 1/N$),

$$S_q = k \frac{N^{1-q}-1}{1-q} \tag{3}$$

which recovers, for $q=1$, the Boltzmann expression $S = k_B \ln N$. The generalized equilibrium canonical distribution is given by [1, 2]

$$p_i = \frac{[1-\beta(1-q)\varepsilon_i]^{1/(1-q)}}{Z_q} \tag{4a}$$

with

$$Z_q \equiv \sum_{i=1}^{N} [1-\beta(1-q)\varepsilon_i]^{1/(1-q)} \tag{4b}$$

(with ε_i the energy of the ith microstate and $\beta \equiv 1/kT$, T being the temperature).

This generalization of Boltzmann–Gibbs statistics leads to a thermodynamics [2] which naturally recovers the standard one in the $q \to 1$ limit.

The generalized statistics has recently been applied with success [3] to the study of stellar systems. Specifically, it overcomes the well known Boltzmann–Gibbs statistics inability for giving *finite* mass in the polytropic spheres model (as discussed by Chandrasekhar and others). In other words, the non-extensive character of the Tsallis entropy seems to be an essential ingredient for recovering *finite* mass, as physically expected.

Since then, the generalized statistics have been the subject of much work, and other generalizations followed (see [4]–[13] (in [13] another application of the Tsallis entropy was found: q was connected with the fractal dimension associated with the d-dimensional Levy flights)).

Following this line, our purpose here is to relate the fluctuations of appropriate variables in the generalized statistics to their corresponding generalized 'susceptibilities'. We intend to generalize, in this way, the fluctuation–dissipation theorem, which plays an important role in the theory of thermodynamic fluctuations (see [14]). We recall briefly the standard relations which lead to this theorem.

Consider a macroscopic system in equilibrium interacting with reservoirs associated to the extensive variables $\hat{X}_0, \hat{X}_1, \ldots, \hat{X}_s$ (which undergo continual macroscopic fluctuations) and restrictive with respect to the extensive variables $X_{s+1}, \ldots, X_t$ (which remain constant). The circumflex above X_k indicates that $\hat{X}_k$ is an instantaneous fluctuating value. The intensive parameter F_k characterizing the reservoir associated to X_k is [14]

$$F_k = \frac{\partial \hat{S}^r}{\partial X_k^r} \tag{5}$$

(r *indicates the reservoir*).

The probability that $\hat{X}_0$ will be found in the range $d\hat{X}_0$, that $\hat{X}_1$ will be found in the range $d\hat{X}_1 \ldots$ and that $\hat{X}_s$ will be found in the range $d\hat{X}_s$ is defined as $W\, d\hat{X}_0\, d\hat{X}_1 \ldots d\hat{X}_s$, where W is the statistical distribution function for the fluctuating variables. As is well known [14], the functional form of W is postulated to be

$$W = \Omega_0 \exp \frac{1}{k_B}\left(\hat{S} - \sum_{k=0}^{s} F_k \hat{X}_k - S[F_0, \ldots, F_S]\right) \tag{6}$$

where k_B is the Boltzmann constant, $\hat{S}$ is the 'instantaneous entropy' [14] of the system and $S[F_0, \ldots, F_s]$ is the maximum value of $\hat{S} - \Sigma_{k=0}^{s} F_k \hat{X}_k$. In [14] it is argued that $S[F_0, \ldots, F_s]$ is identical to the Legendre transform of the *equilibrium* entropy; this argument also leads to the useful relation

$$\frac{\partial S}{\partial F_k} = -X_k. \tag{7}$$

Ω_0 is a normalizing constant such that $\int W(\hat{X}_0, \ldots, \hat{X}_s)\, d\hat{X}_0\, d\hat{X}_1 \ldots d\hat{X}_s = 1$.

In this way the average (*equilibrium*) value of $\hat{X}_k$ is

$$X_k = \int \hat{X}_k W(\hat{X}_0, \ldots, \hat{X}_s)\, \mathrm{d}\hat{X}_0\, \mathrm{d}\hat{X}_1 \ldots \mathrm{d}\hat{X}_s \tag{8}$$

(notice that this is a *temporal* average).

If $\delta\hat{X}_k$ denotes the deviation of $\hat{X}_k$ from X_k,

$$\delta\hat{X}_k \equiv (\hat{X}_k - X_k) \tag{9}$$

a typical second moment of the distribution W is written

$$\langle \delta\hat{X}_j\, \delta\hat{X}_k \rangle = \int (\delta\hat{X}_j\, \delta\hat{X}_k) W(\hat{X}_0, \ldots, \hat{X}_s)\, \mathrm{d}\hat{X}_0\, \mathrm{d}\hat{X}_1 \ldots \mathrm{d}\hat{X}_s. \tag{10}$$

In [14] it is demonstrated that the functional form of the distribution function W gives rise to the fluctuation–dissipation theorem, which reads (for the second moments)

$$\langle \delta\hat{X}_j\, \delta\hat{X}_k \rangle = -k_{\mathrm{B}} \left(\frac{\partial X_j}{\partial F_k}\right) F_0, \ldots, F_{k-1}, F_{k+1}, \ldots, F_s, X_{s+1}, \ldots, X_t \tag{11}$$

where $\partial X_j / \partial F_k$ is the 'susceptibility' of X_j under the action of the 'force' F_k.

2. The generalized fluctuation–dissipation theorem

Now we will look for a suitable generalization of equations (6)–(11).

A crucial point is the generalization of equation (6) for the statistical distribution function W. W can be seen as the probability density of an instantaneous macrostate, which is proportional to the number N of microstates associated with this macrostate. From equation (3) we obtain

$$N = \left[1 + (1-q)\frac{S_q}{k}\right]^{1/(1-q)} \tag{12}$$

which generalizes the relation $N = \mathrm{e}^{S/k_{\mathrm{B}}}$ valid for $q=1$. Then we consistently propose for W_q the following functional form:

$$W_q = \Omega_{0q}\left[1 + \frac{1-q}{k}\left(\hat{S}_q - \sum_{k=0}^{s} F_k \hat{X}_k - S_q[F_0, \ldots, F_s]\right)\right]^{1/(1-q)} \tag{13}$$

where, similarly as before, $\hat{S}_q$ generalizes the 'instantaneous entropy', $F_k = \partial\hat{S}_q^r / \partial X_k^r$ and $S_q[F_0, \ldots, F_s]$ is the maximum value of $\hat{S}_q - \Sigma_{k=0}^{s} F_k \hat{X}_k$. Ω_{0q} is determined by the normalization condition

$$\int W_q\, \mathrm{d}\hat{X}_0\, \mathrm{d}\hat{X}_1 \ldots \mathrm{d}\hat{X}_s = 1. \tag{14}$$

Let us now introduce the 'q-average' of the extensive thermodynamic variable $\hat{X}_k$:

$$X_{kq} \equiv \langle \hat{X}_k \rangle_q \equiv \int \hat{X}_k W_q^q\, \mathrm{d}\hat{X}_0 \ldots \mathrm{d}\hat{X}_s \tag{15}$$

which can also be written as

$$X_{kq}\equiv\langle W_q^{q-1}\hat{X}_k\rangle_1\equiv\int W_q^{q-1}\hat{X}_k W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s. \tag{16}$$

This form enlightens the q-exponent used in equation (15), since in the q-generalization the natural variable is $W_q^{q-1}\hat{X}_k\equiv\hat{X}_{kq}$ [2, 6, 11, 15]. Therefore, it is natural to define

$$\delta\hat{X}_{kq}\equiv W_q^{q-1}\hat{X}_k-X_{kq}. \tag{17}$$

Notice then that

$$\langle\delta\hat{X}_{kq}\rangle_1=\int(\delta\hat{X}_{kq})W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s=0. \tag{18}$$

We also notice that if the important variable is $\hat{X}_{kq}\equiv\hat{X}_k W_q^{q-1}$, then the generalization of equation (7) is written as

$$\frac{\partial S_q}{\partial F_k}=-\frac{X_{kq}}{W_q^{q-1}}. \tag{19}$$

Now we focus our attention on the following second moment of W_q:

$$\langle\delta\hat{X}_{jq}\,\delta\hat{X}_{kq}\rangle_1=\int\delta\hat{X}_{jq}\,\delta\hat{X}_{kq}W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s \tag{20}$$

To carry out this integration we first observe that equations (13) and (19) imply

$$\begin{aligned}\frac{\partial W_q}{\partial F_k}&=\frac{\Omega_{0q}^{1-q}}{k}W_q^q\left(-\hat{X}_k-\frac{\partial S_q[F_0,\ldots,F_s]}{\partial F_k}\right)\\&=-\frac{\Omega_{0q}^{1-q}}{k}W_q(W_q^{q-1}\hat{X}_k-X_{kq})\end{aligned} \tag{21}$$

and using equation (17),

$$\frac{\partial W_q}{\partial F_k}=-\frac{\Omega_{0q}^{1-q}}{k}W_q\,\delta\hat{X}_{kq}. \tag{22}$$

Equation (20) can now be written as

$$\langle\delta\hat{X}_{jq}\,\delta\hat{X}_{kq}\rangle_1=-\frac{k}{\Omega_{0q}^{1-q}}\int\delta\hat{X}_{jq}\frac{\partial W_q}{\partial F_k}\delta\hat{X}_0\ldots\mathrm{d}\hat{X}_s \tag{23}$$

hence

$$\langle\delta\hat{X}_{jq}\,\delta\hat{X}_{kq}\rangle_1=-\frac{k}{\Omega_{0q}^{1-q}}\left(\frac{\partial}{\partial F_k}\int\delta\hat{X}_{jq}W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s-\int\frac{\partial(\delta\hat{X}_{jq})}{\partial F_k}W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s\right). \tag{24}$$

The first integral vanishes since $\langle\delta\hat{X}_{jq}\rangle_1=0$. Then we have

$$\begin{aligned}\langle\delta\hat{X}_{jq}\,\delta\hat{X}_{kq}\rangle_1&=\frac{k}{\Omega_{0q}^{1-q}}\int\left(\frac{\partial(W_q^{q-1}\hat{X}_j)}{\partial F_k}-\frac{\partial X_{jq}}{\partial F_k}\right)W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s\\&=\frac{k}{\Omega_{0q}^{1-q}}\int\frac{\partial W_q^{q-1}}{\partial F_k}\hat{X}_jW_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s-\frac{k}{\Omega_{0q}^{1-q}}\frac{\partial X_{jq}}{\partial F_k}\end{aligned} \tag{25}$$

using equation (14) and the fact that the fluctuating value $\hat{X}_j$ does not depend on F_k.

Evaluating the derivative $\partial W_q^{q-1}/\partial F_k$ we obtain

$$\frac{\partial W_q^{q-1}}{\partial F_k}=\frac{\Omega_{0q}^{1-q}}{k}\, W_q^{q-1}(1-q)\delta\hat{X}_{kq} \tag{26}$$

and the integral in equation (25) can be written as

$$\begin{aligned}\frac{k}{\Omega_{0q}^{1-q}}\int\frac{\partial W_q^{q-1}}{\partial F_k}\hat{X}_j W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s\\ &=(1-q)\int\delta\hat{X}_{kq}W_q^{q-1}\hat{X}_j W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s\\ &=(1-q)\int\delta\hat{X}_{kq}\hat{X}_{jq}W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s.\end{aligned} \tag{27}$$

Subtracting from equation (27) the integral $(1-q)\int\delta\hat{X}_{kq}X_{jq}W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s$ (which is zero because of equation (18)), equation (25) becomes

$$\begin{aligned}\langle\delta\hat{X}_{jq}\delta\hat{X}_{kq}\rangle_1&=(1-q)\left(\int\delta\hat{X}_{kq}\delta\hat{X}_{jq}W_q\,\mathrm{d}\hat{X}_0\ldots\mathrm{d}\hat{X}_s\right)-\frac{k}{\Omega_{0q}^{1-q}}\frac{\partial X_{jq}}{\partial F_k}\\ &=(1-q)\langle\delta\hat{X}_{jq}\delta\hat{X}_{kq}\rangle_1-\frac{k}{\Omega_{0q}^{1-q}}\frac{\partial X_{jq}}{\partial F_k}.\end{aligned} \tag{28}$$

Finally, we have

$$q\langle\delta\hat{X}_{jq}\delta\hat{X}_{kq}\rangle_1=-\frac{k}{\Omega_{0q}^{1-q}}\left(\frac{\partial X_{jq}}{\partial F_k}\right)F_0,\ldots,F_{k-1},F_{k+1},\ldots,F_s \tag{29}$$

which is the generalized form of the fluctuation–dissipation theorem (equation (11)).

We remark that in the simple case $\hat{X}_k=\hat{X}_j\equiv\hat{X}$, the left-hand side of equation (29), $\langle(\delta\hat{X}_q)^2\rangle_1$, measures the magnitude of the fluctuations of $\hat{X}_q\equiv W_q^{q-1}\hat{X}$. Recalling that $\delta\hat{X}_q=W_q^{q-1}\hat{X}-X_q$ ($X_q\equiv\langle\hat{X}\rangle_q$) and since $\langle\hat{X}\rangle_q=\langle W_q^{q-1}\hat{X}_1\rangle$, we can write

$$\begin{aligned}\langle(\delta\hat{X}_q)^2\rangle_1&=\langle(W_q^{q-1})^2\hat{X}\hat{X}-W_q^{q-1}\hat{X}X_q-X_qW_q^{q-1}\hat{X}+X_qX_q\rangle_1\\ &=\langle(W_q^{q-1}\hat{X})^2\rangle_1-\langle W_q^{q-1}\hat{X}\rangle_1^2\end{aligned} \tag{30}$$

or alternatively

$$\langle(\delta\hat{X}_q)^2\rangle_1=\left\langle\frac{\hat{X}^2}{W_q^{1-q}}\right\rangle_q-\langle\hat{X}\rangle_q^2. \tag{31}$$

3. Applications

We will now illustrate expression (29) for two different situations, namely for the specific heat of a thermal system [16] and for the susceptibility of a magnetic system in the presence of an external uniform magnetic field H [17] (C Tsallis, personal communication).

In [16] it was shown that the generalized specific heat is given by the relation

$$C_q \equiv T\frac{\partial S_q}{\partial T} = \frac{\partial U_q}{\partial T} \tag{32}$$

where U_q is the generalized internal energy [2]. Using the equations (4*a*) and (4*b*) for the canonical distribution functions, and by performing *ensemble* averages, da Silva *et al* obtained the following expression for C_q:

$$\frac{C_q}{k} = \frac{q}{(kT)^2}\left[\left(\sum_{i=1}^{N} p_i^q \frac{\varepsilon_i^2}{1-\beta(1-q)\varepsilon_i}\right) - \left(\sum_{i=1}^{N} p_i^q \varepsilon_i\right)\left(\sum_{i=1}^{N} p_i \frac{\varepsilon_i}{1-\beta(1-q)\varepsilon_i}\right)\right] \tag{33}$$

where ε_i and p_i are the quantities already defined. Equation (33) can be alternatively written as

$$\frac{C_q}{k} = \frac{q}{(kT)^2} Z_q^{q-1}\left[\left(\sum_i p_i \frac{\varepsilon_i^2}{(p_i^{1-q})^2}\right) - \left(\sum_i p_i \frac{\varepsilon_i}{p_i^{1-q}}\right)^2\right] \tag{34}$$

since $p_i^q = p_i/p_i^{1-q}$ and $1-\beta(1-q)\varepsilon_i = (p_i Z_q)^{1-q}$ (from equation (4*a*)). Inspecting equation (34) we recognize a natural variable, $\varepsilon/[p(\varepsilon)]^{1-q}$, and $q=1$ *ensemble* averages corresponding to the fluctuation of this variable. Equation (34) can then be written as

$$q\left\langle\left[\delta\left(\frac{\varepsilon}{(p(\varepsilon))^{1-q}}\right)\right]^2\right\rangle_1 = \frac{kT^2}{Z_q^{q-1}}\frac{\partial U_q}{\partial T}. \tag{35}$$

Now to compare this expression with equation (29) we must identify temporal and ensemble averages, i.e. we must assume the ergodic hypothesis. In this way, $U_q = \langle\hat{\varepsilon}\rangle_q$, $\hat{\varepsilon}$ being an instantaneous value of the energy of the system. The normalization condition (equation (14)) and ergodicity imply that

$$\Omega_{0q} = \frac{1}{Z_q}. \tag{36}$$

We also see that the fluctuations which appear in equations (35) and (29) are identical averages, i.e.

$$\left\langle\left[\delta\left(\frac{\hat{\varepsilon}}{W(\hat{\varepsilon})^{1-q}}\right)\right]^2\right\rangle_1 = \left\langle\left[\delta\left(\frac{\varepsilon}{p(\varepsilon)^{1-q}}\right)\right]^2\right\rangle_1. \tag{37}$$

Recalling that the reservoir parameter associated with U_q is $F=1/T$,we finally identify equation (35) as a particular case of equation (29).

As a second example, we consider a magnetic system of N spins S_i in an external uniform magnetic field H. In this system, the generalized isothermal susceptibility is defined as

$$\chi_q \equiv \left.\frac{\partial M_q}{\partial H}\right|_{H=0} \tag{38}$$

where $M_q \equiv \langle S^z\rangle_q \equiv \langle\Sigma_{i=1}^N S_i^z\rangle_q$. Tsallis [17] (C Tallis, personal communication) proved that

$$\chi_q = \frac{q}{kT}\left(\left\langle\frac{S^{z2}}{(p_n Z_q)^{1-q}}\right\rangle_q - \langle S^z\rangle_q\left\langle\frac{S^z}{(p_n Z_q)^{1-q}}\right\rangle_1\right) \tag{39}$$

or equivalently

$$q\left(\left\langle \frac{S^{z^2}}{(p_n^{1-q})^2} \right\rangle_1 - \left\langle \frac{S^z}{p_n^{1-q}} \right\rangle_1^2\right) = \frac{kT}{Z_q^{q-1}} \frac{\partial M_q}{\partial H}. \tag{40}$$

In a similar way to that of the first example, the above expression is readily identified with a particular case of equation (29). In the present system, $M_q = \langle \hat{S} \rangle_q$, $\hat{S}$ being an instantaneous value of the magnetization of the system. The reservoir parameter associated with M_q is $F = -H/T$, and then $\partial M_q/\partial F = T\partial M_q/\partial H$ (for an isothermal transformation).

4. Conclusion

In summary, we have derived a generalization of the fluctuation–dissipation theorem using the Tsallis statistics [1]. A crucial step is the generalization of the statistical distribution function W which describes the *macroscopic fluctuations* and averages of any thermodynamic quantity of a system in equilibrium in contact with reservoirs. We recover as particular cases of the theorem the generalized relations already obtained for the specific heat in terms of the fluctuations of the total energy and for the magnetic susceptibility in terms of the fluctuations of the total magnetization. They both reproduce the well known results in the limit $q = 1$.

Acknowledgments

We would like to thank Professor Constantino Tsallis for valuable discussions and for providing us with [15] and [17] and Dr Thadeu Penna for calling our attention to this problem. Partial support form CNPq and FINEP is also acknowledged.

References

[1] Tsallis C 1988 *J. Stat. Phys.* **52** 479
[2] Curado E M F and Tsallis C 1991 *J. Phys. A: Math. Gen.* **24** L69 (errata: 1991 *J. Phys. A: Math. Gen.* **24** 3187; 1992 *J. Phys. A: Math. Gen* 1019)
[3] Plastino A R and Plastino A 1993 *Phys. Lett.* **174A** 384
[4] Mariz A M 1992 *Phys. Lett.* **165A** 409
[5] Ramshaw J D 1993 *Phys. Lett.* **175A** 169; 1993 *Phys. Lett.* **175A** 171
[6] Plastino A R and Plastino A 1983 *Phys. Lett.* **177A** 177
[7] Plastino A and Tsallis C 1993 *J. Phys. A: Math. Gen.* **26** L893
[8] Plastino A R and Plastino A 1994 *Physica* **202-A** 438
[9] Stariolo D A 1994 *Phys. Lett.* **185A** 262
[10] Silver R N and Tsallis C 1993 Generalized entropy as a possible basis for a nonlinear Schroedinger-like equation *Preprint*, CBPF (Centro Brasiliero de Pesquises Fisicas)
[11] Sarmento E F 1993 Generalization of single-site Callen's identity within Tsallis statistics *Preprint*
[12] Buyukkilic F and Demirhan D 1993 *Phys. Lett.* **181A** 24
[13] Alemany P A and Zanette T H 1993 Fractal random walks from a variational formalism for Tsallis entropies *Phys. Rev.* E **49** 956
[14] Callen H B 1960 *Thermodynamics* (Chichester: Wiley)
Callen H B and Welton T A 1951 *Phys. Rev.* **83** 34

[15] Tsallis C 1993 Non extensive physics: a possible connection between generalized statistical mechanisms and quantum groups *Preprint*, CBPF
[16] da Silva E P, Tsallis C and Curado E M F 1993 *Physica* **199A** 137
[17] Tsallis C 1994 Extensive Versus Non-extensive Physics *New Trends in Magnetic Materials and their Applications* ed J L Morán–Lopez and J M Sanchez (New York: Plenum) in press